Mechanism design without money

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1 Introduction

We consider the facility location problem. For example, let there be n agents living along the real line. We want to build a public facility to serve them. The agents are asked to report their locations and each agent would like to minimize the cost from his location to the facility. The goal of the mechanism is to place a facility (or in general k facilities). The goal of the mechanism could be to minimize the social cost, which is the sum of the cost for each agent or to minimize the maximum cost that any agent experience. The cost of each agent is the distance to his nearest facility.

In general, optimal mechanisms need not be strategy proof. The inability to use money (or rather payments) in many situations of practical relevance like public goods location or political decision making further makes the problem challenging [4]. In this context, 'Approximate Mechanism Design without Money' looks to trade efficiency for strategyproofness in cases where payments are not possible. Approximation is the tool employed to enable strategyproofness and the goal of the design process is to come up with tight approximate mechanisms.

1.1 The setting

We have N={1,2,...,n} agents located on a line and each agent i has a location x_i. The mechanism $f : \Re^n \to \Re^k$ maps the reported locations to k facility locations. Let $\{y_1, .., y_k$ be the locations of the facilities. Then the cost for each agent is

$$cost(f(\mathbf{x}, x_i)) = \min_{j=1,...k} |y_j - x_i|$$

The social cost is defined as

$$SC(f(\mathbf{x}, \mathbf{x})) = \sum_{i=1}^{n} cost(f(\mathbf{x}, x_i))$$

and the maximum cost incurred by any agent is

$$MC(f(\mathbf{x},\mathbf{x})) = \max_{i=1\ldots n} cost(f(\mathbf{x},x_i))$$

These expressions can be extended for randomized mechanisms also. In that case the output of the an allocation rule f would be distributions over $\Re^k$ and we would consider *cost*, MC and SC in expectations. In the following sections we review the literature on facility location that minimizes MC and SC on a line.

1.2 1 Facility

First let us consider the Social Cost. In order to avoid the impossibility result given by the Gibbard-Satterthwaite theorem, we restrict ourselves to single peaked preferences. The optimal and strategyproof allocation for 1 facility is the median location among those reported $\mathbf{x}$. This is also a group strategyproof mechanism.

Next we consider the Maximum Cost (MC). The optimal allocation is to place the facility at the mean of the facilities at left and right extremes. This is clearly not a strategyproof mechanism. If payments were allowed, we could have run a VCG mechanism which is strategyproof and optimal. In our case, with no payments we resort to approximate mechanisms. In [3], the authors show that and order statistic, say choosing the left i^{th} reported location is a 2 approximate group strategyproof deterministic mechanism. Further this bound is tight.

Randomization improves the approximation and we can get a 3/2 approximate group strategyproof mechanism [3]. This mechanism returns the left extreme point and right extreme point with probability 1/4 each. The mean of the two extremes is returned with probability 1/2.

Thus for 1 facility, median is a optimal strategyproof mechanism for SC. For MC, randomization leads to a 3/2 approximate algorithm as compared to 2 approximate for deterministic cases.

1.3 2 Facilities

1.3.1 Maximum Cost

We use the following notation. $\mathrm{lt}(\mathbf{x})$ and $\mathrm{rt}(\mathbf{x})$ denote the left and right extreme point of the bid vector $\mathbf{x}$. $\{lt(\mathbf{x}), rt(\mathbf{x})\}$ is a 2 approximate, group strategyproof mechanism for MC. A corresponding lower bound of 2 can be shown for $n \geq 3$ which makes this bound tight.

Randomization improves the approximation ratio. Define

$lb(\mathbf{x}) = max\{x_i : x_i \leq center(\mathbf{x})\}$, $rb(\mathbf{x}) = min\{x_i : x_i \geq center(\mathbf{x})\}$ and

$$dist(\mathbf{x}) = max\{lb(\mathbf{x}) - lt(\mathbf{x}), rt(\mathbf{x}) - rb(\mathbf{x})\}$$

The meachanism which returns $\{lt(\mathbf{x}), rt(\mathbf{x})\}$ with probability 1/2 , $\{lt(\mathbf{x}) + dist(\mathbf{x}), rt(\mathbf{x}) - dist(\mathbf{x})\}$ with probability 1/6 and $\{lt(\mathbf{x}) + dist(\mathbf{x})/2, rt(\mathbf{x}) - dist(\mathbf{x})/2\}$ with probability 1/3 is a 5/3 approximate strategy proof mechanism [3]. Proving truthfulness of the mechanism is is done by enumerating all possible cases. Whether this mechanism is group strategyproof or not is unknown. Also, the best known lower bound ([3]) is 3/2.

To summarize, deterministic mechanism gives a 2 approximate algorithm where as for randomized mechanisms, we have an upper bound of 5/3 and a lower bound of 3/2.

1.3.2 Social Cost

First we look at deterministic mechanisms. The simplest mechanism to choose two facilities deterministically is $\{lt(\mathbf{x}), rt(\mathbf{x})\}$. This is strategyproof, because the the agents at the end points do not have an incentive to misreport. Further any agent between them can only pust the facility further away by becoming the left or tight extreme. This is an $n-2$ approximate mechanism and the worst performance is when the reported locations is of the form $\{0, 0.5, 0.5, \ldots, 0.5, 1\}$. A corresponding lower bound of a 2 approximate mechanism was shown in [2] which was further improved in [1] to $\frac{n-1}{2}$. Thus $n-2$ and $\frac{n-1}{2}$ are the best known upper and lower bounds for deterministic mechanisms.

Let us consider a simple randomized mechanism described in [2]. Place the facilities at $\{lt(\mathbf{x}), rt(\mathbf{x})\}$ with probability 1/2 and at $\{lt(\mathbf{x}) + dist(\mathbf{x}), rt(\mathbf{x}) - dist(\mathbf{x})\}$ with probability 1/2 This is a strategyproof $n/2$ approximate mechanism. This gives an upper bound as $n/2$ for a randomized strategyproof mechanism. Further, a lower bound of 1.045 was established. A constant factor approximation was obtained in [2] which we describe and improve in the next section.

2 Proportional Mechanism

The proportional mechanism places the facilities in two stages. Denote y_1 and y_2 as the location of the first and the second facility.

- In the first stage the facility is places by choosing a reported location uniformly at random. Thus the probability of a the facility being located at x_i is $\frac{1}{n}, \forall i$.

- In the second stage, the facility is placed randomly with a probability proportional to the distance from the first facility. If the first facility is placed at y_1, denote $d(y_1, x_i)$ as the distance of agent i from the facility y_1. Thus the probability of the second facility being located at j is $\frac{d(y_1,x_j)}{\sum_i d(y_1,x_i)}$

We first show that the mechanism is strategyproof.

Proof: Use the notation $cost_k(f(\mathbf{x}), x_i)$ as the expected cost for agent i when the first facility is located at k. By using the conditional expectation rule,

$$cost(f(\mathbf{x}), x_i) = \frac{1}{n}\sum_{k=1}^{n} cost_k(f(\mathbf{x}), x_i)$$

Suppose agent i misreports his loction from x_i to x_i'. The location vector is now $(x_i', \mathbf{x}_{-i})$. We show a much stronger statement

$$cost_k(f(\mathbf{x}'), x_i) \geq cost_k(f(\mathbf{x}), x_i) \quad \forall k \neq i$$

which means the there is no incentive for an agent to misreport even if the first facility location is known. Let $d_i = d(y_1, x_i) = d(x_k, x_i)$ and $d_i' = d(y_1, x_i) = d(x_k, x_i)$. The cost for an agent i is

$$cost_k(f(\mathbf{x}), x_i) = \frac{\sum_{j\neq i} d_j min\{d_i, d(x_i, x_j)\}}{\sum_{j=1}^{n} d_j}$$

The cost for the misreporting agent, written in terms of the truthful cost is

$$cost_k(f(\mathbf{x}'), x_i) = \frac{cost_k(f(\mathbf{x}), x_i)\sum_{j=1}^{n} d_j}{\sum_{j=1}^{n} d_j + (d_i' - d_i)} + \frac{d_i' min\{d_i', d(x_i, x_i')\}}{\sum_{j=1}^{n} d_j + (d_i' - d_i)} \tag{1}$$

Suppose $d_i' \leq d_i$. Then the first term in the rhs of (1) is greater than $cost_k(f(\mathbf{x}), x_i)$ and the second term is non negative. Now we consider the case of $d_i' > d_i$. Combining the two terms in the rhs of (1), we find the following condition sufficient to show strategyproofness

$$d_i' min\{d_i, d(x_i, x_i')\} - (d_i' - d_i) cost_k(f(\mathbf{x}), x_i) \geq 0 \tag{2}$$

If $min\{d_i, d(x_i, x_i')\} = d_i$ then $d_i' \geq d_i' - d_i$. Also the i^{th} agent can choose the first facility located at x_k so that $d_i \geq cost_k(f(\mathbf{x}), x_i)$. Hence (2) holds.

If $min\{d_i, d(x_i, x_i')\} = d(x_i, x_i')$ then $d_i' \geq d_i \geq cost_k(f(\mathbf{x}), x_i)$ and $d(x_i, x_i') \geq d_i' - d_i$. Thus (2) holds.

This proportional mechanism can not be extended to three facilities [1]. In this extension we place the third facility randomly with probability proportional to minimum distance from the other two. Suppose there are a large number of agents located at 0 (meaning that the first facility is located at 0 almost surey), 50 agents located at 1, 4 agents located at $1 + 10^5$ and 1 agent located at $101 + 10^5$. We see that the agent located at 1 has an incentive to misreport his location to $1 + 10^5$. This increases the probability of the the facility being closer to him. Thus the 3 facility location via a proportional mechanism is not startegyproof.

References

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