Stability and control of switching network models

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Network models

"A network is a set of items, which we call nodes, with connections between them, called edges"[1]

- Networks models are widely used in several domains
- Eg: Spread of epidemics, opinions on social networks, failure in infrastructure networks
- Models are characterized by
  - Nodes
  - Network topology (adjacency matrix $A = [a_{ij}]$)
  - A description of the process (How it evolves in time)

[1] *M Newman, The structure ans function of complex networks*

Network topology may change with time

- Eg: Long term models, demand cycles in infrastructure networks, noise
- Deterministic or random transitions

- Our work:
  - Model for process on networks with randomly switching topologies
  - Stability
  - Control

I. Model formulation

Model

- N nodes
- $x(t) \in \mathbb{R}^{N \times 1}$ is the state node at time t
- M discrete modes $\{m_1, m_2, \ldots, m_M\}$
  - Each corresponds to an adjacency matrix A_m

Model

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- $x(t) \in \mathbb{R}^{N\times 1}$ is the state node at time t
- M discrete modes $\{m_1, m_2, \ldots, m_M\}$
  - Each corresponds to an adjacency matrix A_m
- Linear system dynamics within each mode:

$$x(t+1) = f(A_m)x(t)$$

  - For simplicity, $x(t+1) = \Gamma_m x(t)$

Model (continued)

Node state : $x(t)$
Discrete mode: m (finite set)
Linear dynamics: $x(t+1) = \Gamma_m x(t)$

- Modes (and the topology) change with time
  - Markov transition

$$\mathbb{P}[m(t+1) = j | m(t) = i] = \pi_{ij}(t)$$

  - Transition matrix Π_t are known for a system
  - Note: Transition matrices may be time varying

Summary of Model

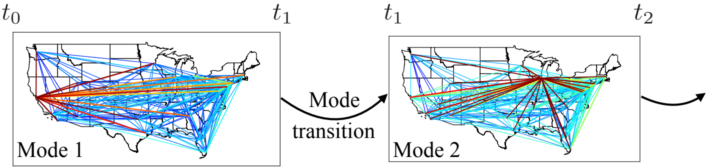
- The system is completely described by

$$x(t+1) = \Gamma_{m(t)} x(t)$$
$$\mathbb{P}[m(t+1) = j | m(t) = i] = \pi_{ij}(t)$$
$$x(0), m(0) \quad \text{(given)}$$

- Features: Continuous node representation, weighted directed interaction and switching topology
- This is a (Positive) Markov Jump Linear System

Model example: Air traffic delay

- Nodes: Airports, $x(t)$ delay level at airports
- 12 discrete modes (Eg: Chicago inc, SFO dec, Low NAS dec ..)
- Transition probability Π_t depends on the time of day



$\vec{x}(t+1) = \Gamma_1 \vec{x}(t)$ $\qquad$ $\vec{x}(t+1) = \Gamma_2 \vec{x}(t)$

- Model is built and validated using operational data [2]

[2] ACC 16, CDC 16

II. Stability of the switched network model

What does stability of MJLS mean?

- For a classic LTI system $x(t+1) = Ax(t)$, stability depends on spectral radius of A

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- For a classic LTI system $x(t+1) = Ax(t)$, stability depends on spectral radius of A
- However stability of Γ_1, ..., Γ_M is neither necessary nor sufficient
- $x(t)$ is a random variable (need formal stability definition)
- Why do we care about stability?
  - If unstable, want to control
  - If stable, want to recover optimally

Asymptotic stability definitions

How does $x(t)$ evolve as $t \to \infty$?

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Mean Stability:
For any $\vec{x}(0) \geq 0$, $\lim_{t\to\infty} \mathbb{E}[||\vec{x}(t)||] = 0$

Asymptotic stability definitions

How does $x(t)$ evolve as $t \to \infty$?

Mean Stability:
For any $\vec{x}(0) \geq 0$, $\lim_{t\to\infty} \mathbb{E}[||\vec{x}(t)||] = 0$

Almost Surely Stable:
For any $\vec{x}(0) \geq 0$, $\mathbb{P}[\lim_{t\to\infty} ||\vec{x}(t)|| = 0] = 1$

Stability Results

System dynamics:

$$x(t+1) = \Gamma_{m(t)} x(t)$$
$$\mathbb{P}[m(t+1) = j | m(t) = i] = \pi_{ij}$$
$$x(t) \geq 0 \quad \forall t$$
$$x(0), m(0) \quad \text{(given)}$$

Result 1

The system is Mean Stable if and only if

$$\text{Spectral Radius}(\mathcal{B}) < 1$$

where $\mathcal{B} = (\Pi^T \otimes \mathbb{I}_n) diag(\Gamma_i)$

Stability Results

System dynamics:

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$$\mathbb{P}[m(t+1) = j | m(t) = i] = \pi_{ij}(t)$$
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$$x(0), m(0) \quad \text{(given)}$$

Result 1 (extension)

If Markovian transition matrices are periodic with time period K, the system is mean stable if and only iff

$$\text{Spectral Radius } (\mathcal{B}_k \mathcal{B}_{k+1} \ldots \mathcal{B}_{k+K}) < 1$$

for some $k \in [0, K]$ and $\mathcal{B}_t = (\Pi_t^T \otimes \mathbb{I}_n) diag(\Gamma_i)$

Stability Results

System dynamics:

$$
\begin{aligned}
&x(t+1) = \Gamma_{m(t)} x(t) \\
&\mathbb{P}[m(t+1) = j | m(t) = i] = \pi_{ij}(t) \\
&x(t) \geq 0 \quad \forall t \\
&x(0), m(0) \quad \text{(given)}
\end{aligned}
$$

Result 2

Mean Stability $\Rightarrow$ Almost Sure Stability

Example: Air Traffic Delay Model

- Periodic system (24 hours) and time dependent transition probability, i.e $\Pi = \Pi_t$,
  - Spectral radius $(\mathcal{B}_1 \mathcal{B}_2 \dots \mathcal{B}_{24}) = 0.67 < 1$
  - Mean Stable and Almost Surely Stable

Example: Air Traffic Delay Model

- Periodic system (24 hours) and time dependent transition probability, i.e $\Pi = \Pi_t$,
  - Spectral radius $(\mathcal{B}_1\mathcal{B}_2 \dots \mathcal{B}_{24}) = 0.67 < 1$
  - Mean Stable and Almost Surely Stable

- If average transition matrix for the day,
  - Spectral radius $(\mathcal{B}) = 1.061 > 1$
  - Not Mean Stable

The temporal patterns in the discrete mode transition probabilities are critical in stabilizing the system

III. Control of the switched network model

What can we control?

Aim: Minimize $||\sum_{t=1}^{T}\mathbb{E}[x(t)]||$ (eg. delay cost)

Two control options:

1. Discrete mode
   - Can force certain mode transitions to occur
   - Modifies the transition probability Π
   - Penalty on forced transition

What can we control?

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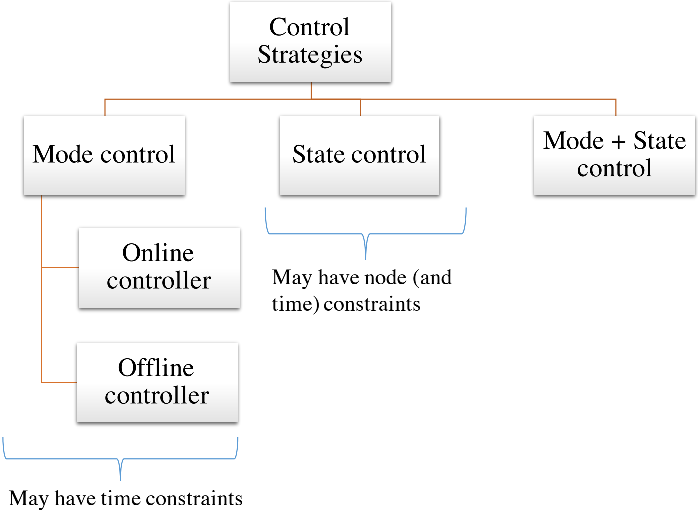
Two control options:

1. Discrete mode
   - Can force certain mode transitions to occur
   - Modifies the transition probability Π
   - Penalty on forced transition
2. Continuous state
   - $x(t+1) = \Gamma_{m(t)} x(t) - u(t)$
   - Penalty of $\beta^T u(t)$

Overview


Control Strategies
Mode control
State control
Mode + State control
Online controller
Offline controller
May have node (and time) constraints
May have time constraints

Mode controller - Online

- Observe $x(t^*)$, $m(t^*)$ at time t^*
- Control decision: Force a transition or random transition (Π_t)
- Penalty: $C(i, j, t)$ if forced from mode i to j at t

- Minimize: $\sum_{t=t^*}^{T} ||\mathbb{E}[x(t)]|| + \sum_{i,j} C(m(t^*), j)\mathbf{1}_{forced}$

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- Minimize: $\sum_{t=t^*}^{T} ||\mathbb{E}[x(t)]|| + \sum_{i,j} C(m(t^*), j)\mathbf{1}_{forced}$

- Greedy myopic solution:

$$\min\{\ J(x(t^*), i, t^*)\ ,\quad \min_j(\ C(i, j, t^*) + J(\Gamma_i x(t^*), j, t^* + 1)\ )\}$$

$J(x(t^*), i, t^*) = \sum_{t=t^*}^{T} ||\mathbb{E}[x(t)]||$: Easy to compute

Mode controller - Offline

- Observe: $m(t)$ at time t
- Pre-determined control decision: Force a transition or random transition (Π_t)
- Minimize $\sum_{t=1}^{T} ||\mathbb{E}[x(t)]|| + \mathbb{E}[\sum_{i,j,t} C(i,j,t)]$

Mode controller - Offline

- Observe: $m(t)$ at time t
- Pre-determined control decision: Force a transition or random transition (Π_t)
- Minimize $\sum_{t=1}^{T} ||\mathbb{E}[x(t)]|| + \mathbb{E}[\sum_{i,j,t} C(i,j,t)]$
- Solution: Greedy heuristic

Step 1 Compute cost for all forced transitions (i, j, t)
Step 2 Add min cost transition to control policy and update Π
Step 3 Repeat until cost stops decreasing

State controller

- Observe: $x(t^*)$ and $m(t^*)$
- Myopic, greedy formulation to decide on $u(t^*)$
- Penalty: $\beta^T u(t^*)$
- Optimize $u(t^*)$ at time t^*

State controller

- Observe: $x(t^*)$ and $m(t^*)$
- Myopic, greedy formulation to decide on $u(t^*)$
- Penalty: $\beta^T u(t^*)$
- Optimize $u(t^*)$ at time t^*

$$\begin{aligned}
\text{Min} \quad & \sum_{t=t^*}^{T} ||\mathbb{E}[x(t)]|| + \beta^T u(t^*) \\
\text{s.t} \quad & x(t^*+1) = \Gamma_{m(t^*)} x(t^*) - u(t^*) \\
& \vdots \qquad\qquad \vdots \\
& x(t^*+1) \geq 0, \quad 0 \leq u(t^*) \leq U_{max}
\end{aligned}$$

State controller - Solution

- Threshold policy

$$u_i(t^*) = \begin{cases} \min\{U_{max}, [\Gamma_{m(t^*)} x(t^*)]_i\} & \text{if } \beta_i < d_i \\ 0 & \text{otherwise} \end{cases}$$

d_i is a known function (depends on Π_t and Γ_m)

- Can account for temporal constraints and node constraints easily

Application: Air traffic delay model

- Baseline, no control: $||\sum_{t=1}^{24}\mathbb{E}[x(t)]|| = 1$
- Mode control (expected delay $+$ mode transition cost)
  - Online heuristic: 0.86
  - Offline heuristic: 0.88

Application: Air traffic delay model

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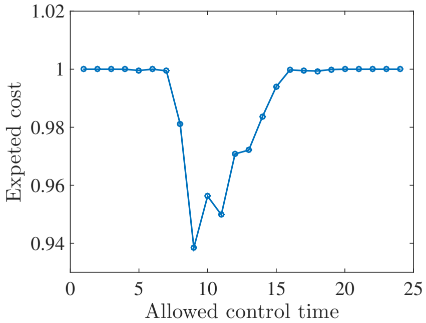
- Mode control (expected delay + mode transition cost)
  - Online heuristic: 0.86
  - Offline heuristic: 0.88
  - Offline heuristic with temporal constraint: 0.94



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  - Cost with control at all airports: 0.17

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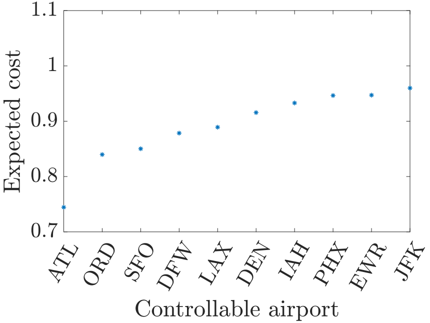
- State control (expected delay + state control cost)
  - Cost with control at all airports: 0.17
  - Cost with control at only one airport:



Summary

1. Switched system model for switching topology network models
2. Conditions for the system to be stable
3. Heuristics to control the mode and state

Extensions:

- Finite time stability
- Optimal mode controllers, integration of mode and state control